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# Two-Step Stability Analysis for General Polynomial-Fuzzy-Model-Based Control Systems

H.K. Lam, *Senior Member, IEEE*, Ligang Wu, *Senior Member, IEEE* and James Lam *Fellow, IEEE*

**Abstract**—This paper investigates the stability of polynomial-fuzzy-model-based (PFMB) control system formed by a nonlinear plant represented by a polynomial fuzzy model and a polynomial fuzzy controller connected in a closed loop. Three cases of polynomial fuzzy controllers are proposed for the control process with the consideration of matched/mismatched number of rules and/or premise membership functions, which demonstrate different levels of controller complexity, design flexibility and stability analysis results. A general polynomial Lyapunov function candidate is proposed to investigate the system stability. Unlike the published work, there is no constraint on the polynomial Lyapunov function candidate, which is independent of the form of the polynomial fuzzy model. Thus, it can be applied to a wider class of PFMB control systems and potentially produces more relaxed stability analysis result. Two-step stability conditions in terms of sum-of-squares (SOS) are obtained to find numerically a feasible solution. To facilitate the stability analysis and relax the stability analysis result, the boundary information of membership functions is taken into account in the stability analysis and incorporated into the SOS-based stability conditions. Simulation examples are given to illustrate the effectiveness of the proposed approach.

**Index Terms**—Fuzzy-Model-Based Control, Mismatched Premise Membership Functions, Nonlinear Systems, Polynomial Fuzzy Systems, Stability Analysis, Sum-of-Squares.

## I. INTRODUCTION

**F**UZZY control offers an alternative control approach for nonlinear systems. In the past decades, in general, two categories of fuzzy control approaches, namely model-free [1]–[3] and fuzzy-model-based (FMB) [4] control, have been proposed. Model-free control approach does not require a mathematical model for the control design. The design of the fuzzy controller is mainly based on the knowledge on the system and through a heuristic process. Under the model-free control approach, it suffers from two drawbacks: 1) it does not guarantee the stability of the overall fuzzy control system and 2) it is difficult to conduct system analysis. These drawbacks motivate the development of FMB control approach

which offers a systematic control and analysis techniques for nonlinear plants. Under the FMB control paradigm, the T-S fuzzy model [5], [6] offers a systematic way to represent the dynamics of the nonlinear plants supporting the stability analysis and control synthesis. The most popular type of fuzzy controllers is the state-feedback fuzzy controller (referred to as fuzzy controller hereafter), which is employed to close the feedback loop for the control process. Other fuzzy controllers such as adaptive fuzzy controller [7]–[18], decentralized fuzzy controller [19], fuzzy sliding-mode controller [20]–[27], fuzzy controller with fault-tolerant design [28], fuzzy controller for time-delay systems [29],  $H_\infty$  fuzzy controller [30], [31], output-feedback fuzzy controller [32], switching fuzzy controller [33]–[39], sampled-data fuzzy controller [40]–[45] and 2-D fuzzy controller [46] can also be found in the literature.

In general, there are three categories of FMB control systems according to the number of rules and premise membership functions. The first category is that the fuzzy controller does not share the same number of rules and premise membership functions as those of the T-S fuzzy model, which is in favor of the design flexibility. When a smaller number of rules and/or simple membership functions are employed, it can reduce the controller complexity resulting in a lower implementation cost. Basic stability conditions in terms of linear matrix inequalities (LMIs) were obtained in [47], [48] to determine the system stability and control synthesis. Convex programming techniques [49] can be used to find numerically a feasible solution to the LMI-based stability conditions.

The second category is that the fuzzy controller shares the same number of rules and premise membership functions as those of the T-S fuzzy model, which is also known as the parallel distributed compensation (PDC) design concept [47], [48]. Because of the perfectly matched membership functions, it is in favor of the stability analysis resulting in more relaxed stability conditions by grouping the same multiplication terms of membership functions. However, on the contrary to the first category, the PDC design concept does not offer any design flexibility and will lead to a higher controller complexity especially when the T-S fuzzy model has a large number of rules and/or complex membership functions. LMI-based stability conditions for this category of FMB control systems were obtained in [47], [48], [50]–[56]. With the consideration of the information of membership functions, further relaxed LMI-based stability conditions were obtained in [57]–[60].

The third category is that the fuzzy controller shares the same number of rules but not the premise membership functions. This category can be viewed as a compromise of the first and the second ones. The matched number of rules allows the

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H.K. Lam is with the Department of Informatics, King's College London, Strand, London, WC2R 2LC, United Kingdom. e-mail: hak-keung.lam@kcl.ac.uk.

L. Wu is with Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, 150001, P. R. China. e-mail: ligangwu@hit.edu.cn.

J. Lam is with the Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong. e-mail: james.lam@hku.hk.

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PDC-based stability analysis approach to be used in a certain level for achieving relaxed LMI-based stability conditions (compared with the first category). The freedom of choosing the premise membership functions offers a greater design flexibility to fuzzy controller. By choosing simple membership functions, the implementation cost can be reduced and lower compared with that in the second category. LMI-based stability conditions for this category were obtained in [61]–[66].

Recently, the T-S fuzzy model has been extended to polynomial fuzzy model [67]–[69]. With the consideration of polynomials, it enhances the system modeling capability and thus the polynomial fuzzy model is able to represent a wider class of nonlinear plants. However, it will end up with stability conditions depending on the state variables of the system such that LMI solver cannot be used to find a feasible solution numerically. Instead, sum-of-squares (SOS) approach [70] was then employed to investigate the stability of polynomial-fuzzy-model-based (PFMB) control systems [67], [68]. Based on the Lyapunov stability theory, basic PDC SOS-based stability conditions [67], [68] were obtained. A feasible solution to the SOS-based stability conditions can be found numerically using the third-party Matlab toolbox SOSTOOLS [71]. Relaxed stability analysis results can be found in [69], [72]–[75]. The work in [69] was based on PDC design concept of which the technique of variable transformation was employed for the stability analysis. The work in [72]–[75] considered the non-PDC design concept of which the technique of membership function approximation was employed for the stability analysis.

In this paper, we shall investigate the stability of PFMB control systems of all three categories based on the SOS-based approach combining with the Lyapunov stability theory. The drawback of the existing SOS-based stability analysis is that the polynomial Lyapunov function candidate depends on the form of the polynomial fuzzy model. In order to widen the applicability of the PFMB control approach, we eliminate the aforementioned limitation of the polynomial Lyapunov function candidate such that its polynomial matrix can be dependent on any state variables. SOS-based stability conditions are obtained with the consideration of the boundary information of membership functions to determine the system stability and facilitate the control synthesis. A two-step procedure is proposed to find numerically a feasible solution to the proposed SOS-based stability conditions.

This paper is organized as follows. In Section II, notations used in this paper are introduced. The details of the polynomial fuzzy model and polynomial fuzzy controller are presented. In Section III, SOS-based stability conditions are obtained for the PFMB control systems of the three categories based on the Lyapunov stability theory. A two-step procedure is proposed to find numerically a feasible solution. In Section IV, simulation examples are given to illustrate the merits of the proposed PFMB control scheme. In section V, a conclusion is drawn.

## II. NOTATION AND PRELIMINARIES

### A. Notation

Throughout the paper, the following notations are adopted [70]. A monomial in  $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]$  is a function

of the form  $x_1^{d_1}(t) \cdots x_n^{d_n}(t)$  where  $d_i$ ,  $i = 1, \dots, n$ , are nonnegative integers. The degree of a monomial is defined as  $d = \sum_{i=1}^n d_i$ . A polynomial  $\mathbf{p}(\mathbf{x}(t))$  is defined as a finite linear combination of monomials with real coefficients. A polynomial  $\mathbf{p}(\mathbf{x}(t))$  is a sum of squares if it can be written as  $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^m \mathbf{q}_j(\mathbf{x}(t))^2$  where  $\mathbf{q}_j(\mathbf{x}(t))$  is a polynomial and  $m$  is a non-zero positive integer. Hence, it can be seen that  $\mathbf{p}(\mathbf{x}(t)) \geq 0$  if it is an SOS. The expressions of  $\mathbf{M} > 0$ ,  $\mathbf{M} \geq 0$ ,  $\mathbf{M} < 0$  and  $\mathbf{M} \leq 0$  denote the positive, semi-positive, negative and semi-negative definite matrices  $\mathbf{M}$ , respectively.

### B. Polynomial Fuzzy Model

Let  $p$  be the number of fuzzy rules describing the behavior of a nonlinear plant [67], [68]. The  $i$ -th rule is of the following format:

Rule  $i$ : IF  $f_1(\mathbf{x}(t))$  is  $M_1^i$  AND  $\cdots$  AND  $f_\Psi(\mathbf{x}(t))$  is  $M_\Psi^i$   
THEN  $\dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)$  (1)

where  $M_\alpha^i$  is the fuzzy term of rule  $i$  corresponding to the function  $f_\alpha(\mathbf{x}(t))$ ,  $\alpha = 1, \dots, \Psi$ ;  $i = 1, \dots, p$ ;  $\Psi$  is a positive integer;  $\mathbf{x}(t) \in \mathbb{R}^n$  is the system state vector;  $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times n}$  and  $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$  are the known polynomial system and input matrices, respectively;  $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \mathbb{R}^N$  is a vector of monomials in  $\mathbf{x}(t)$ ;  $\mathbf{u}(t) \in \mathbb{R}^m$  is the input vector. It is assumed that  $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$  if and only if  $\mathbf{x}(t) = \mathbf{0}$ . The system dynamics is described as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)), \quad (2)$$

where  $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$ ,  $w_i(\mathbf{x}(t)) \geq 0 \forall i$  and  $w_i(\mathbf{x}(t)) = \prod_{l=1}^\Psi \mu_{M_l^i}(f_l(\mathbf{x}(t))) / \sum_{k=1}^p \prod_{l=1}^\Psi \mu_{M_l^k}(f_l(\mathbf{x}(t))) \quad \forall i$ ,  $w_i(\mathbf{x}(t))$  is the normalized grade of membership;  $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ ,  $\alpha = 1, \dots, \Psi$ , is the grade of membership corresponding to the fuzzy term  $M_\alpha^i$ .

### C. Polynomial Fuzzy Controller

A polynomial fuzzy controller described by the following  $c$  rules is introduced to stabilize the nonlinear plant represented by the polynomial fuzzy model (2).

Rule  $j$ : IF  $g_1(\mathbf{x}(t))$  is  $N_1^j$  AND  $\cdots$  AND  $g_\Omega(\mathbf{x}(t))$  is  $N_\Omega^j$   
THEN  $\mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t))$ , (3)

where  $N_\beta^j$  is the fuzzy term of rule  $j$  corresponding to the function  $g_\beta(\mathbf{x}(t))$ ,  $\beta = 1, \dots, \Omega$ ;  $j = 1, \dots, c$ ;  $\Omega$  is a positive integer;  $\mathbf{G}_j(\mathbf{x}(t)) \in \mathbb{R}^{m \times N}$ ,  $j = 1, \dots, c$ , is the polynomial feedback gain to be determined. The polynomial fuzzy controller is defined as follows:

$$\mathbf{u}(t) = \sum_{j=1}^c m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \quad (4)$$

where  $\sum_{j=1}^c m_j(\mathbf{x}(t)) = 1$ ,  $m_j(\mathbf{x}(t)) \geq 0 \forall j$ ,  $m_j(\mathbf{x}(t)) = \prod_{l=1}^\Omega \mu_{N_l^j}(g_l(\mathbf{x}(t))) / \sum_{k=1}^c \prod_{l=1}^\Omega \mu_{N_l^k}(g_l(\mathbf{x}(t))) \quad \forall j$ ,  $m_j(\mathbf{x}(t))$  is

the normalized grade of membership;  $\mu_{N_\alpha^j}(g_\alpha(\mathbf{x}(t)))$ ,  $\beta = 1, \dots, \Omega$ , is the grade of membership corresponding to the fuzzy term  $N_\beta^j$ .

### III. STABILITY ANALYSIS

A PFMB control system is formed by the polynomial fuzzy model (2) and polynomial fuzzy controller (4) connected in a closed loop. From (2) and (4), using the property of membership functions that  $\sum_{i=1}^p w_i(\mathbf{x}(t)) = \sum_{j=1}^c m_j(\mathbf{x}(t)) = \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) = 1$ , the PFMB control system is obtained as follows:

$$\begin{aligned} \dot{\mathbf{x}}(t) = & \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t)) \\ & + \mathbf{B}_i(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t)))\hat{\mathbf{x}}(\mathbf{x}(t)) \end{aligned} \quad (5)$$

The control objective is to determine the polynomial feedback gains  $\mathbf{G}_j(\mathbf{x}(t))$  such that the PFMB system (5) is asymptotically stable, that is,  $\mathbf{x}(t) \rightarrow \mathbf{0}$  as time  $t \rightarrow \infty$ .

We shall investigate the stability of the PFMB system (5) using the SOS-based approach with the support of the Lyapunov stability theory. SOS-based stability conditions will be obtained to determine the system stability and facilitate the control synthesis. In the following analysis, for brevity, the time  $t$  associated with the variables is dropped for the situation without ambiguity, e.g.,  $\mathbf{x}(t)$  and  $\mathbf{u}(t)$  are denoted as  $\mathbf{x}$  and  $\mathbf{u}$ , respectively. Furthermore,  $\hat{\mathbf{x}}(\mathbf{x}(t))$ ,  $w_i(\mathbf{x}(t))$  and  $m_j(\mathbf{x}(t))$  are denoted as  $\hat{\mathbf{x}}$ ,  $w_i$  and  $m_j$ , respectively.

To proceed with the stability analysis, we denote  $\mathbf{x} = [x_1, \dots, x_n]^T$  and  $\hat{\mathbf{x}} = [\hat{x}_1, \dots, \hat{x}_N]^T$ . From (5), we have

$$\begin{aligned} \dot{\hat{\mathbf{x}}} = & \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}} \\ = & \sum_{i=1}^p \sum_{j=1}^c w_i m_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))\hat{\mathbf{x}}, \end{aligned} \quad (6)$$

where  $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$ ,  $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$  and

$$\mathbf{T}(\mathbf{x}) = \left[ \frac{\partial \hat{x}_i(\mathbf{x})}{\partial x_j} \right]_{i=1, \dots, N; j=1, \dots, n}. \quad (7)$$

Because of the assumption  $\hat{\mathbf{x}} = \mathbf{0}$  if and only if  $\mathbf{x} = \mathbf{0}$ , the stability of the PFMB control system (6) implies that of (5).

Three cases of FPMB control system are considered, namely, Case 1:  $c = p$  and  $m_i = w_i$  for all  $i$ , Case 2:  $c = p$  and  $m_i \neq w_i$  for any  $i$ , and Case 3:  $c \neq p$  and  $w_i$  and  $m_j$  are different for any  $i$  and  $j$ .

#### A. Case 1: $c = p$ and $m_i = w_i$ for all $i$

In this case, the polynomial fuzzy controller shares the same number of rules and membership functions of the T-S fuzzy model, which is in favor of the stability analysis using the property of perfectly matched premises and PDC-based analysis approach [47], [48].

We consider the following polynomial Lyapunov function candidate to investigate the system stability of (6),

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T \mathbf{X}(\mathbf{x})^{-1} \hat{\mathbf{x}}, \quad (8)$$

where  $0 < \mathbf{X}(\mathbf{x}) = \mathbf{X}(\mathbf{x})^T \in \Re^{N \times N}$  is a polynomial matrix to be determined. From (6) and (8), with  $c = p$  and  $m_i = w_i$  for all  $i$ , we have,

$$\begin{aligned} \dot{V}(\mathbf{x}) = & \sum_{i=1}^p \sum_{j=1}^p w_i w_j \hat{\mathbf{x}}^T \left( (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))^T \mathbf{X}(\mathbf{x})^{-1} \right. \\ & \left. + \mathbf{X}(\mathbf{x})^{-1} (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})) \right) \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \left[ \frac{d\mathbf{X}(\mathbf{x})^{-1}}{dt} \right] \hat{\mathbf{x}}. \end{aligned} \quad (9)$$

Define  $\mathbf{z} = \mathbf{X}(\mathbf{x})^{-1} \hat{\mathbf{x}}$  and  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1}$  where  $\mathbf{N}_j(\mathbf{x}) \in \Re^{m \times N}$ ,  $j = 1, \dots, c$ , is an arbitrary polynomial matrix. From (9), we have

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^p \sum_{j=1}^p w_i w_j \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x}) \mathbf{z}, \quad (10)$$

where  $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\mathbf{x}) + \mathbf{X}(\mathbf{x})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k=1}^n \frac{\partial \mathbf{X}(\mathbf{x})}{\partial x_k} (\mathbf{A}_i^k(\mathbf{x}) + \mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1})\hat{\mathbf{x}}$  for  $i = 1, \dots, p$ ;  $j = 1, \dots, c$ ;  $\mathbf{A}_i^k(\mathbf{x}) \in \Re^N$  and  $\mathbf{B}_i^k(\mathbf{x}) \in \Re^m$  denote the  $k^{th}$  row of  $\mathbf{A}_i(\mathbf{x})$  and  $\mathbf{B}_i(\mathbf{x})$ , respectively.

We introduce the boundary information of membership functions [57], [58], [60]–[66] through some slack polynomial matrices to the stability analysis for relaxing the stability conditions. Introducing the slack matrices  $0 \leq \underline{\mathbf{R}}_{ij}(\mathbf{x}) \in \Re^{N \times N}$  and  $0 \leq \bar{\mathbf{R}}_{ij}(\mathbf{x}) \in \Re^{N \times N}$ , we have

$$\sum_{i=1}^p \sum_{j=1}^p (w_i w_j - \underline{\gamma}_{ij}) \underline{\mathbf{R}}_{ij}(\mathbf{x}) \geq 0, \quad (11)$$

$$\sum_{i=1}^p \sum_{j=1}^p (\bar{\gamma}_{ij} - w_i w_j) \bar{\mathbf{R}}_{ij}(\mathbf{x}) \geq 0, \quad (12)$$

where  $\underline{\gamma}_{ij} = \underline{\gamma}_{ji}$  and  $\bar{\gamma}_{ij} = \bar{\gamma}_{ji}$  are the lower and upper bounds, respectively, of  $w_i w_j$  satisfying  $\underline{\gamma}_{ij} \leq w_i w_j \leq \bar{\gamma}_{ij}$  for all  $i$  and  $j$ . From (10), (11) and (12), we have

$$\dot{V}(\mathbf{x}) \leq \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p w_i w_j \mathbf{z}^T (\Phi_{1ij}(\mathbf{x}) + \Phi_{1ji}(\mathbf{x})) \mathbf{z}, \quad (13)$$

where  $\Phi_{1ij}(\mathbf{x}) = \mathbf{Q}_{ij}(\mathbf{x}) + \underline{\mathbf{R}}_{ij}(\mathbf{x}) - \bar{\mathbf{R}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{R}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \underline{\mathbf{R}}_{rs}(\mathbf{x}))$ .

Based on the Lyapunov stability theory,  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) imply the asymptotic stability of (6) which can be achieved by

$$\mathbf{X}(\mathbf{x}) > 0, \quad (14)$$

$$\underline{\mathbf{R}}_{ij}(\mathbf{x}) \geq 0 \quad \forall i, j, \quad (15)$$

$$\bar{\mathbf{R}}_{ij}(\mathbf{x}) \geq 0 \quad \forall i, j, \quad (16)$$

$$\Phi_{1ij}(\mathbf{x}) + \Phi_{1ji}(\mathbf{x}) < 0 \quad \forall i, j. \quad (17)$$

However, because of the term  $\mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1}$  in  $\mathbf{Q}_{ij}(\mathbf{x})$ , the condition  $\Phi_{1ij}(\mathbf{x}) + \Phi_{1ji}(\mathbf{x}) < 0$  is not convex and thus convex programming techniques cannot be employed to find numerically a feasible solution. Instead, a two-step procedure is proposed in this paper to search for a feasible solution.

To develop the two-step procedure, we consider another polynomial Lyapunov function candidate to investigate the stability of the PFMB control system (6) as follows.

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \hat{\mathbf{x}}, \quad (18)$$

where  $\mathbf{P}(\mathbf{x}) = \mathbf{P}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$  is a polynomial matrix which is chosen such that  $\hat{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \hat{\mathbf{x}} > 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ).

From (5) and (18), we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \hat{\mathbf{x}}^T \dot{\mathbf{P}}(\mathbf{x}) \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \dot{\hat{\mathbf{x}}} + \hat{\mathbf{x}}^T \dot{\mathbf{P}}(\mathbf{x}) \hat{\mathbf{x}} \\ &= \sum_{i=1}^p \sum_{j=1}^p w_i w_j \hat{\mathbf{x}}^T \mathbf{H}_{ij}(\mathbf{x}) \hat{\mathbf{x}}, \end{aligned} \quad (19)$$

where  $\mathbf{H}_{ij}(\mathbf{x}) = (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}))^T \mathbf{P}(\mathbf{x}) + \mathbf{P}(\mathbf{x}) (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x})) + \sum_{k=1}^n \frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_k} (\mathbf{A}_i^k(\mathbf{x}) + \mathbf{B}_i^k(\mathbf{x}) \mathbf{G}_j(\mathbf{x})) \hat{\mathbf{x}}$  for  $i, j = 1, \dots, p$ .

We introduce the slack polynomial matrices  $\underline{\mathbf{S}}_{ij} \in \mathbb{R}^{N \times N}$  and  $\bar{\mathbf{S}}_{ij} \in \mathbb{R}^{N \times N}$ , satisfying  $\hat{\mathbf{x}}^T \underline{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0$  and  $\hat{\mathbf{x}}^T \bar{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0$ , respectively, for all  $i$  and  $j$ , through the following inequalities to the stability analysis.

$$\sum_{i=1}^p \sum_{j=1}^p (w_i w_j - \gamma_{ij}) \hat{\mathbf{x}}^T \underline{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0, \quad (20)$$

$$\sum_{i=1}^p \sum_{j=1}^p (\bar{\gamma}_{ij} - w_i w_j) \hat{\mathbf{x}}^T \bar{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0, \quad (21)$$

Adding (20) and (21) to (19), we have

$$\dot{V}(\mathbf{x}) \leq \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p w_i w_j \hat{\mathbf{x}}^T (\boldsymbol{\Theta}_{1ij}(\mathbf{x}) + \boldsymbol{\Theta}_{1ji}(\mathbf{x})) \hat{\mathbf{x}}, \quad (22)$$

where  $\boldsymbol{\Theta}_{1ij}(\mathbf{x}) = \mathbf{H}_{ij}(\mathbf{x}) + \underline{\mathbf{S}}_{ij}(\mathbf{x}) - \bar{\mathbf{S}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{S}}_{rs}(\mathbf{x}) - \gamma_{rs} \underline{\mathbf{S}}_{rs}(\mathbf{x}))$ .

Based on the Lyapunov stability theory,  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) imply the asymptotic stability of (6) which can be achieved by

$$\hat{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \hat{\mathbf{x}} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}, \quad (23)$$

$$\hat{\mathbf{x}}^T \underline{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (24)$$

$$\hat{\mathbf{x}}^T \bar{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (25)$$

$$\hat{\mathbf{x}}^T (\boldsymbol{\Theta}_{1ij}(\mathbf{x}) + \boldsymbol{\Theta}_{1ji}(\mathbf{x})) \hat{\mathbf{x}} < 0 \quad \forall \mathbf{x} \neq \mathbf{0}, i, j. \quad (26)$$

However, because of the term  $\mathbf{G}_j(\mathbf{x})$  in  $\mathbf{H}_{ij}(\mathbf{x})$ , the condition  $\hat{\mathbf{x}}^T (\boldsymbol{\Theta}_{1ij}(\mathbf{x}) + \boldsymbol{\Theta}_{1ji}(\mathbf{x})) \hat{\mathbf{x}} < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) is not convex and thus convex programming techniques cannot be employed to find numerically a feasible solution.

*Remark 1:* The stability conditions (14) to (17) are in the form of  $\mathbf{M}(\mathbf{x}) > 0$  (because  $\mathbf{z}$  is independent of  $\mathbf{x}$ ), where  $\mathbf{M}(\mathbf{x})$  is a polynomial matrix, while the stability conditions (23) to (26) are in the form of  $\hat{\mathbf{x}}^T \mathbf{M}(\mathbf{x}) \hat{\mathbf{x}}$ . Comparing the two sets of stability conditions, as  $\mathbf{M}(\mathbf{x}) > 0$  implies  $\hat{\mathbf{x}}^T \mathbf{M}(\mathbf{x}) \hat{\mathbf{x}} > 0$  but not the other way round, the form in (23) to (26) is easier to obtain a feasible solution.

*Remark 2:* It is observed that the two sets of stability conditions (14) to (17) and (23) to (26) are not convex and

convex programming techniques cannot be applied to find numerically a feasible solution. However, when the term  $\mathbf{B}_i^k(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\mathbf{x})^{-1}$  in  $\mathbf{Q}_{ij}(\mathbf{x})$  is ignored and  $\mathbf{G}_j(\mathbf{x})$  in  $\mathbf{H}_{ij}(\mathbf{x})$  is pre-defined, the stability conditions (14) to (17) and (23) to (26) become convex.

A two-step procedure is proposed to find numerically a feasible solution based on the property of the stability conditions in Remark 2. In general, the first step is to ignore the term  $\mathbf{B}_i^k(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\mathbf{x})^{-1}$  in  $\mathbf{Q}_{ij}(\mathbf{x})$  of the stability conditions (14) to (17) and find a feasible solution  $\mathbf{G}_j(\mathbf{x})$ . In the second step, using the obtained  $\mathbf{G}_j(\mathbf{x})$  in the first step, search for a feasible solution of the stability conditions (23) to (26). If there exists a feasible solution in the second step, the PFMB control system (6) is guaranteed to be asymptotically stable.

*Remark 3:* Recall that  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\mathbf{x})^{-1}$  in the first step. If there exists a feasible solution  $\mathbf{X}(\mathbf{x})$ , we have  $\mathbf{X}(\mathbf{x})^{-1} = \frac{\text{adj}(\mathbf{X}(\mathbf{x}))}{\det(\mathbf{X}(\mathbf{x}))}$  where  $\text{adj}(\mathbf{X}(\mathbf{x}))$  and  $\det(\mathbf{X}(\mathbf{x}))$  denote the adjoint and determinant of the matrix  $\mathbf{X}(\mathbf{x})$ , respectively. The term  $\det(\mathbf{X}(\mathbf{x}))$  in the denominator makes convex programming techniques unable to apply. To circumvent the difficulty, the following technique is employed. If  $\mathbf{X}(\mathbf{x}) > 0$ , it implies that  $\det(\mathbf{X}(\mathbf{x})) > 0$ . Multiplying  $\det(\mathbf{X}(\mathbf{x}))$  to (26), we have  $\det(\mathbf{X}(\mathbf{x})) \mathbf{x}^T (\boldsymbol{\Theta}_{1ij}(\mathbf{x}) + \boldsymbol{\Theta}_{1ji}(\mathbf{x})) \mathbf{x} < 0$ . Expanding  $\boldsymbol{\Theta}_{1ij}(\mathbf{x})$  in the stability condition, we obtain  $\det(\mathbf{X}(\mathbf{x})) \boldsymbol{\Theta}_{1ij}(\mathbf{x}) = (\det(\mathbf{X}(\mathbf{x})) \tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \text{adj}(\mathbf{X}(\mathbf{x})))^T \mathbf{P}(\mathbf{x}) + \mathbf{P}(\mathbf{x}) (\det(\mathbf{X}(\mathbf{x})) \tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \text{adj}(\mathbf{X}(\mathbf{x}))) + \sum_{k=1}^n \frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_k} (\det(\mathbf{X}(\mathbf{x})) \mathbf{A}_i^k(\mathbf{x}) + \mathbf{B}_i^k(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \text{adj}(\mathbf{X}(\mathbf{x}))) \hat{\mathbf{x}} + \det(\mathbf{X}(\mathbf{x})) (\underline{\mathbf{S}}_{ij}(\mathbf{x}) - \bar{\mathbf{S}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{S}}_{rs}(\mathbf{x}) - \gamma_{rs} \underline{\mathbf{S}}_{rs}(\mathbf{x})))$  of which  $\mathbf{X}(\mathbf{x})$  and  $\mathbf{N}_j(\mathbf{x})$  are obtained from the stability conditions (14) to (17) in the first step and they are not decision variables in the second step. It can be seen that  $\det(\mathbf{X}(\mathbf{x})) \mathbf{x}^T (\boldsymbol{\Theta}_{1ij}(\mathbf{x}) + \boldsymbol{\Theta}_{1ji}(\mathbf{x})) \mathbf{x} < 0$  becomes convex and convex programming techniques can be applied to search numerically for a feasible solution.

The stability analysis result and detailed two-step procedure are summarized in the following theorem.

*Theorem 1:* The PFMB control system (5) formed by the nonlinear plant represented by the polynomial fuzzy model in the form of (2) and the polynomial fuzzy controller (4) connected in a closed loop, of which  $c = p$  and  $m_i = w_i$  for all  $i$ , is asymptotically stable if there exists a feasible solution to the following two-step procedure:

**First step:** Defining decision polynomial matrices  $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ ,  $\underline{\mathbf{R}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $\bar{\mathbf{R}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\mathbf{X}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $i, j = 1, \dots, p$ , find a feasible solution to the following SOS-based stability conditions:

$$\nu^T (\mathbf{X}(\mathbf{x}) - \varepsilon_1(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS,}$$

$$\nu^T \underline{\mathbf{R}}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$\nu^T \bar{\mathbf{R}}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$-\nu^T(\hat{\Phi}_{1ij}(\mathbf{x}) + \hat{\Phi}_{1ji}(\mathbf{x}) + \varepsilon_2(\mathbf{x})\mathbf{I})\nu \text{ is SOS } \forall i, j,$$

where  $\nu \in \mathbb{R}^N$  is an arbitrary vector independent of  $\mathbf{x}$ ;  $\varepsilon_1(\mathbf{x}) > 0$  and  $\varepsilon_2(\mathbf{x}) > 0$  are predefined scalar polynomials;  $\underline{\gamma}_{ij}$  and  $\bar{\gamma}_{ij}$  are the lower and upper bounds, respectively, of  $w_i w_j$  satisfying  $\underline{\gamma}_{ij} \leq w_i w_j \leq \bar{\gamma}_{ij}$  for all  $i$  and  $j$ ;  $\hat{\Phi}_{1ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k=1}^n \frac{\partial \mathbf{X}(\mathbf{x})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\tilde{\mathbf{x}} + \mathbf{R}_{ij}(\mathbf{x}) - \bar{\mathbf{R}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{R}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \mathbf{R}_{rs}(\mathbf{x}))$  and the polynomial feedback gains are defined as  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$ .

If there exists  $\mathbf{X}(\mathbf{x})$  such that  $\nu^T(\mathbf{X}(\mathbf{x}) - \varepsilon_1(\mathbf{x})\mathbf{I})\nu$  being an SOS and  $\det(\mathbf{X}(\mathbf{x})) \geq \eta(\mathbf{x})$ , where  $\eta(\mathbf{x})$  is a pre-defined scalar polynomial satisfying  $\eta(\mathbf{x}) > 0$ , the following second step will be proceeded, otherwise, no feasible solution is found for the PFMB control system (5).

**Second step:** Defining the decision polynomial matrices  $\mathbf{P}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $\mathbf{S}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\bar{\mathbf{S}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $i, j = 1, \dots, p$ , find a feasible solution to the following SOS-based stability conditions:

$$\hat{\mathbf{x}}^T(\mathbf{P}(\mathbf{x}) - \varphi_1(\mathbf{x})\mathbf{I})\hat{\mathbf{x}} \text{ is SOS,}$$

$$\hat{\mathbf{x}}^T \mathbf{S}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \text{ is SOS } \forall i, j,$$

$$\hat{\mathbf{x}}^T \bar{\mathbf{S}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \text{ is SOS } \forall i, j,$$

$$-\hat{\mathbf{x}}^T(\hat{\Theta}_{1ij}(\mathbf{x}) + \hat{\Theta}_{1ji}(\mathbf{x}) + \varphi_2(\mathbf{x})\mathbf{I})\hat{\mathbf{x}} \text{ is SOS } \forall i, j,$$

where  $\varphi_1(\mathbf{x}) > 0$  and  $\varphi_2(\mathbf{x}) > 0$  are predefined scalar polynomials;  $\hat{\Theta}_{1ij}(\mathbf{x}) = \det(\mathbf{X}(\mathbf{x}))\Theta_{1ij}(\mathbf{x}) = \Xi_{ij}(\mathbf{x}) + \sum_{k=1}^n \frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_k} (\det(\mathbf{X}(\mathbf{x}))\mathbf{A}_i^k(\mathbf{x}) + \mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\text{adj}(\mathbf{X}(\mathbf{x})))\hat{\mathbf{x}} + \det(\mathbf{X}(\mathbf{x}))(\mathbf{S}_{ij}(\mathbf{x}) - \bar{\mathbf{S}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{S}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \mathbf{S}_{rs}(\mathbf{x})))$ ;  $\Xi_{ij}(\mathbf{x}) = (\det(\mathbf{X}(\mathbf{x}))\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})\text{adj}(\mathbf{X}(\mathbf{x})))^T \mathbf{P}(\mathbf{x}) + \mathbf{P}(\mathbf{x})(\det(\mathbf{X}(\mathbf{x}))\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})\text{adj}(\mathbf{X}(\mathbf{x})))$ ;  $\mathbf{X}(\mathbf{x})$  and  $\mathbf{N}_j(\mathbf{x})$  are obtained from the first step.

*Remark 4:* It should be noted that the polynomial Lyapunov function candidates (8) and (18) employ the polynomial matrices  $\mathbf{X}(\mathbf{x})$  and  $\mathbf{P}(\mathbf{x})$  in  $\mathbf{x}$ , respectively. It means that there is no constraint on the state variables to be used in the polynomial Lyapunov functions candidates. Unlike some existing work [67]–[69], [72]–[74],  $\mathbf{X}(\tilde{\mathbf{x}})$  or  $\mathbf{P}(\tilde{\mathbf{x}})$  is used in the polynomial Lyapunov function candidate, where  $\tilde{\mathbf{x}} = [\tilde{x}_{k_1} \dots \tilde{x}_{k_q}]$  and  $\{k_1, \dots, k_q\}$  is defined as the set of row number that the entries of the entire row of  $\mathbf{B}_i(\mathbf{x})$  are all zero for all  $i$ . By getting rid of this constraint, the stability conditions can be applied to a general PFMB control system and the solution space of  $\mathbf{X}(\mathbf{x})$  or  $\mathbf{P}(\mathbf{x})$  can be enlarged resulting in more relaxed stability analysis result.

*Remark 5:* The purpose of introducing  $\eta(\mathbf{x})$  in the two-step procedure is to prevent the values of the coefficients of  $\det(\mathbf{X}(\mathbf{x}))$  from going too small which may affect the solution searching process in the second step.

*B. Case 2:  $c = p$  and  $m_i \neq w_i$  for any  $i$*

In this case, we consider that the fuzzy controller shares the same number of rules but not the premise membership functions as those of the polynomial fuzzy model. It offers a greater design flexibility of the membership functions which can reduce the complexity of the polynomial fuzzy controller by employing simple membership functions resulting in a lower implementation cost compared with Case 1. Also, the matched number of rules is in favor of the stability analysis by using the property of the PDC-based analysis approach.

We consider the polynomial Lyapunov functions candidates (8) and (18) to obtain the SOS-based stability conditions for the first and second steps of the two-step procedure.

Recalling that  $\mathbf{z} = \mathbf{X}(\mathbf{x})^{-1}\hat{\mathbf{x}}$  and  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1}$ , from (6) and (8), with  $c = p$  and  $m_i \neq w_i$  for any  $i$ , we have,

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \sum_{i=1}^p \sum_{j=1}^p w_i w_j \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x}) \mathbf{z} \\ &+ \sum_{i=1}^p \sum_{j=1}^p w_i (m_j - w_j) \mathbf{z}^T \mathbf{J}_{ij}(\mathbf{x}) \mathbf{z}. \end{aligned} \quad (27)$$

where  $\mathbf{J}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T$ .

Consider the lower and upper bounds of  $m_j - w_j$  as  $\underline{\sigma}_j$  and  $\bar{\sigma}_j$ , respectively, such that  $\underline{\sigma}_j \leq m_j - w_j \leq \bar{\sigma}_j$  for all  $j$ ;  $0 \leq \mathbf{W}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\mathbf{W}_{ij}(\mathbf{x}) \geq \mathbf{J}_{ij}(\mathbf{x})$  for all  $i$  and  $j$  resulting in  $(\bar{\sigma}_j - \underline{\sigma}_j)\mathbf{W}_{ij}(\mathbf{x}) \geq (m_j - w_j - \underline{\sigma}_j)\mathbf{J}_{ij}(\mathbf{x})$  for all  $i$  and  $j$ . From (11), (12) and (27), we have

$$\dot{V}(\mathbf{x}) \leq \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p w_i w_j \mathbf{z}^T (\Phi_{2ij}(\mathbf{x}) + \Phi_{2ji}(\mathbf{x})) \mathbf{z}, \quad (28)$$

where  $\Phi_{2ij}(\mathbf{x}) = \mathbf{Q}_{ij}(\mathbf{x}) + \mathbf{R}_{ij}(\mathbf{x}) - \bar{\mathbf{R}}_{ij}(\mathbf{x}) + \sum_{r=1}^p (\sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{R}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \mathbf{R}_{rs}(\mathbf{x})) + \underline{\sigma}_r \mathbf{J}_{ir}(\mathbf{x}) + (\bar{\sigma}_r - \underline{\sigma}_r) \mathbf{W}_{ir}(\mathbf{x}))$ .

Based on the Lyapunov stability theory,  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) imply the asymptotic stability of the PFMB control system (6) which can be achieved by

$$\mathbf{X}(\mathbf{x}) > 0, \quad (29)$$

$$\bar{\mathbf{R}}_{ij}(\mathbf{x}) \geq 0 \forall i, j, \quad (30)$$

$$\bar{\mathbf{R}}_{ij}(\mathbf{x}) \geq 0 \forall i, j, \quad (31)$$

$$\mathbf{W}_{ij}(\mathbf{x}) \geq 0 \forall i, j, \quad (32)$$

$$\mathbf{W}_{ij}(\mathbf{x}) \geq \mathbf{J}_{ij}(\mathbf{x}) \forall i, j, \quad (33)$$

$$\Phi_{2ij}(\mathbf{x}) + \Phi_{2ji}(\mathbf{x}) < 0 \forall i, j. \quad (34)$$

The above stability conditions (29) to (34) will be used to come up with the SOS-based stability conditions in the first step of the two-step procedure.

We then consider the polynomial Lyapunov function candidates (18) for the second step of the two-step procedure. From

(19), with  $c = p$  and  $m_i \neq w_i$  for any  $i$ , we have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \dot{\mathbf{x}} + \dot{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \dot{\mathbf{x}} + \dot{\mathbf{x}}^T \dot{\mathbf{P}}(\mathbf{x}) \dot{\mathbf{x}} \\ &= \sum_{i=1}^p \sum_{j=1}^p w_i w_j \dot{\mathbf{x}}^T \mathbf{H}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \\ &\quad + \sum_{i=1}^p \sum_{j=1}^p w_i (m_j - w_j) \dot{\mathbf{x}}^T \mathbf{K}_{ij}(\mathbf{x}) \dot{\mathbf{x}},\end{aligned}\quad (35)$$

where  $\mathbf{K}_{ij}(\mathbf{x}) = (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}))^T \mathbf{P}(\mathbf{x}) + \mathbf{P}(\mathbf{x}) (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}))$  for  $i = 1, \dots, p$ ;  $j = 1, \dots, c$ .

Similarly, with the introduction of the slack polynomial matrix  $\mathbf{Y}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  satisfying  $\dot{\mathbf{x}}^T \mathbf{Y}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \geq 0$  and  $\dot{\mathbf{x}}^T \mathbf{Y}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \geq \dot{\mathbf{x}}^T \mathbf{K}_{ij}(\mathbf{x}) \dot{\mathbf{x}}$  for all  $i$  and  $j$ , we have  $(\bar{\sigma}_j - \underline{\sigma}_j) \dot{\mathbf{x}}^T \mathbf{Y}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \geq (m_j - w_j - \underline{\sigma}_j) \dot{\mathbf{x}}^T \mathbf{K}_{ij}(\mathbf{x}) \dot{\mathbf{x}}$  for all  $i$  and  $j$ . Adding (20) and (21) to (28), we have

$$\dot{V}(\mathbf{x}) \leq \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p w_i w_j \dot{\mathbf{x}}^T (\boldsymbol{\Theta}_{2ij}(\mathbf{x}) + \boldsymbol{\Theta}_{2ji}(\mathbf{x})) \dot{\mathbf{x}}, \quad (36)$$

where  $\boldsymbol{\Theta}_{2ij}(\mathbf{x}) = \mathbf{H}_{ij}(\mathbf{x}) + \mathbf{S}_{ij}(\mathbf{x}) - \bar{\mathbf{S}}_{ij}(\mathbf{x}) + \sum_{r=1}^p (\sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{S}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \mathbf{S}_{rs}(\mathbf{x})) + \underline{\sigma}_r \mathbf{K}_{ir}(\mathbf{x}) + (\bar{\sigma}_r - \underline{\sigma}_r) \mathbf{Y}_{ir}(\mathbf{x}))$ .

Based on the Lyapunov stability theory,  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) imply the asymptotic stability of the PFMB control system (6) which can be achieved by

$$\dot{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \dot{\mathbf{x}} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}, \quad (37)$$

$$\dot{\mathbf{x}}^T \mathbf{S}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (38)$$

$$\dot{\mathbf{x}}^T \bar{\mathbf{S}}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (39)$$

$$\dot{\mathbf{x}}^T \mathbf{Y}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (40)$$

$$\dot{\mathbf{x}}^T (\mathbf{Y}_{ij}(\mathbf{x}) - \mathbf{K}_{ij}(\mathbf{x})) \dot{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (41)$$

$$\dot{\mathbf{x}}^T (\boldsymbol{\Theta}_{2ij}(\mathbf{x}) + \boldsymbol{\Theta}_{2ji}(\mathbf{x})) \dot{\mathbf{x}} < 0 \quad \forall \mathbf{x} \neq \mathbf{0}, i, j. \quad (42)$$

From the conditions in (29) to (34) and (37) to (42), by ignoring the term  $\mathbf{B}_i^k(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\mathbf{x})^{-1}$  in  $\mathbf{Q}_{ij}(\mathbf{x})$  and with the consideration of Remark 3, similar to Case 1, we obtain the SOS-based stability conditions and two-step procedure.

**Theorem 2:** The PFMB control system (5) formed by the nonlinear plant represented by the polynomial fuzzy model in the form of (2) and the polynomial fuzzy controller (4) connected in a closed loop, of which  $c = p$  and  $m_i \neq w_i$  for any  $i$ , is asymptotically stable if there exists a feasible solution to the following two-step procedure:

**First step:** Defining decision polynomial matrices  $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ ,  $\mathbf{R}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $\bar{\mathbf{R}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\mathbf{X}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $i, j = 1, \dots, p$ , find a feasible solution to the following SOS-based stability conditions:

$$\nu^T (\mathbf{X}(\mathbf{x}) - \varepsilon_1(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS,}$$

$$\nu^T \mathbf{R}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$\nu^T \bar{\mathbf{R}}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$\nu^T \mathbf{W}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$\nu^T (\mathbf{W}_{ij}(\mathbf{x}) - \mathbf{J}_{ij}(\mathbf{x})) \nu \text{ is SOS } \forall i, j,$$

$$-\nu^T (\hat{\boldsymbol{\Phi}}_{2ij}(\mathbf{x}) + \hat{\boldsymbol{\Phi}}_{2ji}(\mathbf{x}) + \varepsilon_2(\mathbf{x}) \mathbf{I}) \nu \text{ is SOS } \forall i, j,$$

where  $\nu \in \mathbb{R}^N$  is an arbitrary vector independent of  $\mathbf{x}$ ;  $\varepsilon_1(\mathbf{x}) > 0$  and  $\varepsilon_2(\mathbf{x}) > 0$  are predefined scalar polynomials;  $\underline{\gamma}_{ij}$  and  $\bar{\gamma}_{ij}$  are the lower and upper bounds, respectively, of  $w_i w_j$  satisfying  $\underline{\gamma}_{ij} \leq w_i w_j \leq \bar{\gamma}_{ij}$  for all  $i$  and  $j$ ;  $\underline{\sigma}_j$  and  $\bar{\sigma}_j$  are the lower and upper bounds of  $m_j - w_j$ , respectively, satisfying  $\underline{\sigma}_j \leq m_j - w_j \leq \bar{\sigma}_j$  for all  $j$ ;  $\mathbf{J}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x}) \mathbf{X}(\mathbf{x}) + \mathbf{X}(\mathbf{x}) \tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T$ ;  $\hat{\boldsymbol{\Phi}}_{1ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x}) \mathbf{X}(\mathbf{x}) + \mathbf{X}(\mathbf{x}) \tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k=1}^n \frac{\partial \mathbf{X}(\mathbf{x})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x}) \dot{\mathbf{x}} + \mathbf{R}_{ij}(\mathbf{x}) - \bar{\mathbf{R}}_{ij}(\mathbf{x}) + \sum_{r=1}^p (\sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{R}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \mathbf{R}_{rs}(\mathbf{x})) + \underline{\sigma}_r \mathbf{J}_{ir}(\mathbf{x}) + (\bar{\sigma}_r - \underline{\sigma}_r) \mathbf{W}_{ir}(\mathbf{x})); and the polynomial feedback gains are defined as  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x}) \mathbf{X}(\mathbf{x})^{-1}$ .$

If there exists  $\mathbf{X}(\mathbf{x})$  such that  $\nu^T (\mathbf{X}(\mathbf{x}) - \varepsilon_1(\mathbf{x}) \mathbf{I}) \nu$  being an SOS and  $\det(\mathbf{X}(\mathbf{x})) \geq \eta(\mathbf{x})$ , where  $\eta(\mathbf{x})$  is a pre-defined scalar polynomial satisfying  $\eta(\mathbf{x}) > 0$ , the following second step will be proceeded, otherwise, no feasible solution is found for the PFMB control system (5).

**Second step:** Defining the decision polynomial matrices  $\mathbf{P}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $\mathbf{S}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\bar{\mathbf{S}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $i, j = 1, \dots, p$ , find a feasible solution to the following SOS-based stability conditions:

$$\dot{\mathbf{x}}^T (\mathbf{P}(\mathbf{x}) - \varphi_1(\mathbf{x}) \mathbf{I}) \dot{\mathbf{x}} \text{ is SOS,}$$

$$\dot{\mathbf{x}}^T \mathbf{S}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \text{ is SOS } \forall i, j,$$

$$\dot{\mathbf{x}}^T \bar{\mathbf{S}}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \text{ is SOS } \forall i, j,$$

$$\dot{\mathbf{x}}^T \mathbf{Y}_{ij}(\mathbf{x}) \dot{\mathbf{x}} \text{ is SOS } \forall i, j,$$

$$\dot{\mathbf{x}}^T (\mathbf{Y}_{ij}(\mathbf{x}) - \mathbf{K}_{ij}(\mathbf{x})) \dot{\mathbf{x}} \text{ is SOS } \forall i, j;$$

$$-\dot{\mathbf{x}}^T (\hat{\boldsymbol{\Theta}}_{2ij}(\mathbf{x}) + \hat{\boldsymbol{\Theta}}_{2ji}(\mathbf{x}) + \varphi_2(\mathbf{x}) \mathbf{I}) \dot{\mathbf{x}} \text{ is SOS } \forall i, j,$$

where  $\varphi_1(\mathbf{x}) > 0$  and  $\varphi_2(\mathbf{x}) > 0$  are predefined scalar polynomials;  $\hat{\boldsymbol{\Theta}}_{2ij}(\mathbf{x}) = \det(\mathbf{X}(\mathbf{x})) \boldsymbol{\Theta}_{2ij}(\mathbf{x}) = \boldsymbol{\Xi}_{ij}(\mathbf{x}) + \sum_{k=1}^n \frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_k} (\det(\mathbf{X}(\mathbf{x})) \mathbf{A}_i^k(\mathbf{x}) + \mathbf{B}_i^k(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \text{adj}(\mathbf{X}(\mathbf{x}))) \dot{\mathbf{x}} + \det(\mathbf{X}(\mathbf{x})) (\mathbf{S}_{ij}(\mathbf{x}) - \bar{\mathbf{S}}_{ij}(\mathbf{x}) + \sum_{r=1}^p (\sum_{s=1}^p (\bar{\gamma}_{rs} \bar{\mathbf{S}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \mathbf{S}_{rs}(\mathbf{x})) + (\bar{\sigma}_r - \underline{\sigma}_r) \mathbf{Y}_{ir}(\mathbf{x}))) + \sum_{r=1}^p \underline{\sigma}_r \bar{\boldsymbol{\Xi}}_{ir}(\mathbf{x}); \boldsymbol{\Xi}_{ij}(\mathbf{x}) = (\det(\mathbf{X}(\mathbf{x})) \tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \text{adj}(\mathbf{X}(\mathbf{x})))^T \mathbf{P}(\mathbf{x}) + \mathbf{P}(\mathbf{x}) (\det(\mathbf{X}(\mathbf{x})) \tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x}) \mathbf{N}_j(\mathbf{x}) \text{adj}(\mathbf{X}(\mathbf{x})));  $\mathbf{X}(\mathbf{x})$  and  $\mathbf{N}_j(\mathbf{x})$  are obtained from the first step.$

### C. Case 3: $c \neq p$

In this case, we consider that the fuzzy controller does not share the same number of rules which implies that the set of premise membership functions is different from that of the fuzzy controller. It offers the greatest design flexibility to the fuzzy controller among the three cases. When a smaller number of rules and/or simple membership functions are employed, the implementation cost can be further reduced compared with Case 2. However, because the property of PDC-based analysis approach cannot be applied, it potentially produces more conservative stability analysis results compared with the previous 2 cases.

We consider the polynomial Lyapunov functions candidates (8) and (18) to obtain the SOS-based stability conditions for the first and second steps of the two-step procedure.

Recalling that  $\mathbf{z} = \mathbf{X}(\mathbf{x})^{-1}\hat{\mathbf{x}}$  and  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1}$ , from (6) and (8), with  $c \neq p$  and  $w_i$  and  $m_j$  are different for any  $i$  and  $j$ , we have,

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^p \sum_{j=1}^c w_i m_j \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x}) \mathbf{z}. \quad (43)$$

With the introduction of the slack polynomial matrices  $0 \leq \underline{\mathbf{T}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $0 \leq \bar{\mathbf{T}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ , we have

$$\sum_{i=1}^p \sum_{j=1}^c (w_i m_j - \underline{\rho}_{ij}) \underline{\mathbf{T}}_{ij}(\mathbf{x}) \geq 0, \quad (44)$$

$$\sum_{i=1}^p \sum_{j=1}^c (\bar{\rho}_{ij} - w_i m_j) \bar{\mathbf{T}}_{ij}(\mathbf{x}) \geq 0, \quad (45)$$

where  $\underline{\rho}_{ij}$  and  $\bar{\rho}_{ij}$  are the lower and upper bounds, respectively, of  $w_i m_j$  satisfying  $\underline{\rho}_{ij} \leq w_i m_j \leq \bar{\rho}_{ij}$  for all  $i$  and  $j$ .

From (43), (44) and (45), we have

$$\dot{V}(\mathbf{x}) \leq \sum_{i=1}^p \sum_{j=1}^c w_i m_j \mathbf{z}^T \Phi_{3ij}(\mathbf{x}) \mathbf{z}. \quad (46)$$

where  $\Phi_{3ij}(\mathbf{x}) = \mathbf{Q}_{ij}(\mathbf{x}) + \underline{\mathbf{T}}_{ij}(\mathbf{x}) - \bar{\mathbf{T}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^c (\bar{\rho}_{rs} \bar{\mathbf{T}}_{rs}(\mathbf{x}) - \underline{\rho}_{rs} \underline{\mathbf{T}}_{rs}(\mathbf{x}))$ .

Based on the Lyapunov stability theory,  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) imply the asymptotic stability of the PFMB control system (6) which can be achieved by

$$\mathbf{X}(\mathbf{x}) > 0, \quad (47)$$

$$\underline{\mathbf{T}}_{ij}(\mathbf{x}) \geq 0 \quad \forall i, j, \quad (48)$$

$$\bar{\mathbf{T}}_{ij}(\mathbf{x}) \geq 0 \quad \forall i, j, \quad (49)$$

$$\Phi_{3ij}(\mathbf{x}) < 0 \quad \forall i, j. \quad (50)$$

The conditions in (47) to (50) will be employed to develop the SOS-based stability conditions in the first step of the two-step procedure.

We consider the polynomial Lyapunov function candidate (18) to develop the SOS-based stability conditions in the second step. Introducing the slack polynomial matrices  $\underline{\mathbf{U}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\bar{\mathbf{U}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  satisfying  $\hat{\mathbf{x}}^T \underline{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0$  and  $\hat{\mathbf{x}}^T \bar{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0$ , we have

$$\sum_{i=1}^p \sum_{j=1}^c (w_i m_j - \underline{\rho}_{ij}) \hat{\mathbf{x}}^T \underline{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0, \quad (51)$$

$$\sum_{i=1}^p \sum_{j=1}^c (\bar{\rho}_{ij} - w_i m_j) \hat{\mathbf{x}}^T \bar{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0, \quad (52)$$

From (19), (51) and (52), we have

$$\dot{V}(\mathbf{x}) \leq \sum_{i=1}^p \sum_{j=1}^c w_i m_j \hat{\mathbf{x}}^T \Theta_{3ij}(\mathbf{x}) \hat{\mathbf{x}}, \quad (53)$$

where  $\Theta_{3ij}(\mathbf{x}) = \mathbf{H}_{ij}(\mathbf{x}) + \underline{\mathbf{U}}_{ij}(\mathbf{x}) - \bar{\mathbf{U}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^c (\bar{\gamma}_{rs} \bar{\mathbf{U}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \underline{\mathbf{U}}_{rs}(\mathbf{x}))$ .

Based on the Lyapunov stability theory,  $V(\mathbf{x}) > 0$  and  $\dot{V}(\mathbf{x}) < 0$  (excluding  $\mathbf{x} = \mathbf{0}$ ) imply the asymptotic stability of the PFMB control system (6) which can be achieved by

$$\hat{\mathbf{x}}^T \mathbf{P}(\mathbf{x}) \hat{\mathbf{x}} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}, \quad (54)$$

$$\hat{\mathbf{x}}^T \underline{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (55)$$

$$\hat{\mathbf{x}}^T \bar{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \geq 0 \quad \forall i, j, \quad (56)$$

$$\hat{\mathbf{x}}^T \Theta_{3ij}(\mathbf{x}) \hat{\mathbf{x}} < 0 \quad \forall \mathbf{x} \neq \mathbf{0}, i, j. \quad (57)$$

Similar to Case 1 and Case 2, from the conditions in (47) to (50) and (54) to (57), by ignoring the term  $\mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1}$  in  $\mathbf{Q}_{ij}(\mathbf{x})$  and with the consideration of Remark 3, we obtain the SOS-based stability conditions and two-step procedure.

**Theorem 3:** The PFMB control system (5) formed by the nonlinear plant represented by the polynomial fuzzy model in the form of (2) and the polynomial fuzzy controller (4) connected in a closed loop, of which  $c \neq p$  and  $w_i$  and  $m_j$  are different for any  $i$  and  $j$ , is asymptotically stable if there exists a feasible solution to the following two-step procedure:

**First step:** Defining decision polynomial matrices  $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ ,  $\underline{\mathbf{T}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $\bar{\mathbf{T}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\mathbf{X}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $i = 1, \dots, p$ ,  $j = 1, \dots, c$ , find a feasible solution to the following SOS-based stability conditions:

$$\nu^T(\mathbf{X}(\mathbf{x}) - \varepsilon_1(\mathbf{x})\mathbf{I})\nu \text{ is SOS,}$$

$$\nu^T \underline{\mathbf{T}}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$\nu^T \bar{\mathbf{T}}_{ij}(\mathbf{x}) \nu \text{ is SOS } \forall i, j,$$

$$-\nu^T(\hat{\Phi}_{3ij}(\mathbf{x}) + \varepsilon_2(\mathbf{x})\mathbf{I})\nu \text{ is SOS } \forall i, j,$$

where  $\nu \in \mathbb{R}^N$  is an arbitrary vector independent of  $\mathbf{x}$ ;  $\varepsilon_1(\mathbf{x}) > 0$  and  $\varepsilon_2(\mathbf{x}) > 0$  are predefined scalar polynomials;  $\underline{\rho}_{ij}$  and  $\bar{\rho}_{ij}$  are the lower and upper bounds, respectively, of  $w_i m_j$  satisfying  $\underline{\rho}_{ij} \leq w_i m_j \leq \bar{\rho}_{ij}$  for all  $i$  and  $j$ ;  $\hat{\Phi}_{3ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k=1}^n \frac{\partial \mathbf{X}(\mathbf{x})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}} + \underline{\mathbf{T}}_{ij}(\mathbf{x}) - \bar{\mathbf{T}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^c (\bar{\gamma}_{rs} \bar{\mathbf{T}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs} \underline{\mathbf{T}}_{rs}(\mathbf{x}))$ ; and the polynomial feedback gains are defined as  $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$ .

If there exists  $\mathbf{X}(\mathbf{x})$  such that  $\nu^T(\mathbf{X}(\mathbf{x}) - \varepsilon_1(\mathbf{x})\mathbf{I})\nu$  being an SOS and  $\det(\mathbf{X}(\mathbf{x})) \geq \eta(\mathbf{x})$ , where  $\eta(\mathbf{x})$  is a pre-defined scalar polynomial satisfying  $\eta(\mathbf{x}) > 0$ , the following second step will be proceeded, otherwise, no feasible solution is found for the PFMB control system (5).

**Second step:** Defining the decision polynomial matrices  $\mathbf{P}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $\underline{\mathbf{U}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$  and  $\bar{\mathbf{U}}_{ij}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ ,  $i = 1, \dots, p$ ,  $j = 1, \dots, c$ , find a feasible solution to the following SOS-based stability conditions:

$$\hat{\mathbf{x}}^T(\mathbf{P}(\mathbf{x}) - \varphi_1(\mathbf{x})\mathbf{I})\hat{\mathbf{x}} \text{ is SOS,}$$

$$\hat{\mathbf{x}}^T \underline{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \text{ is SOS } \forall i, j,$$

$$\hat{\mathbf{x}}^T \bar{\mathbf{U}}_{ij}(\mathbf{x}) \hat{\mathbf{x}} \text{ is SOS } \forall i, j,$$



$$-\hat{\mathbf{x}}^T(\hat{\Theta}_{3ij}(\mathbf{x}) + \varphi_2(\mathbf{x})\mathbf{I})\hat{\mathbf{x}} \text{ is SOS } \forall i, j,$$

where  $\varphi_1(\mathbf{x}) > 0$  and  $\varphi_2(\mathbf{x}) > 0$  are predefined scalar polynomials;  $\hat{\Theta}_{3ij}(\mathbf{x}) = \det(\mathbf{X}(\mathbf{x}))\Theta_{3ij}(\mathbf{x}) = \Xi_{ij}(\mathbf{x}) + \sum_{k=1}^n \frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_k} (\det(\mathbf{X}(\mathbf{x}))\mathbf{A}_i^k(\mathbf{x}) + \mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\text{adj}(\mathbf{X}(\mathbf{x})))\hat{\mathbf{x}} + \det(\mathbf{X}(\mathbf{x}))(\underline{\mathbf{U}}_{ij}(\mathbf{x}) - \bar{\mathbf{U}}_{ij}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^p (\bar{\gamma}_{rs}\bar{\mathbf{U}}_{rs}(\mathbf{x}) - \underline{\gamma}_{rs}\underline{\mathbf{U}}_{rs}(\mathbf{x}));$   $\Xi_{ij}(\mathbf{x}) = (\det(\mathbf{X}(\mathbf{x}))\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})\text{adj}(\mathbf{X}(\mathbf{x})))^T \mathbf{P}(\mathbf{x}) + \mathbf{P}(\mathbf{x})(\det(\mathbf{X}(\mathbf{x}))\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})\text{adj}(\mathbf{X}(\mathbf{x})))$ ;  $\mathbf{X}(\mathbf{x})$  and  $\mathbf{N}_j(\mathbf{x})$  are obtained from the first step.

*Remark 6:* In this paper, the techniques on grouping the terms with the same membership functions in the stability analysis is based on [67]. More relaxed stability analysis results can be obtained by considering other techniques such as the methods of variable transformation [69], [74], approximated membership functions [65], [73] or consideration of various information [75]. However, by using these methods, it will increase the computational demand on solving the solution resulting from the increasing number of variables and SOS-based stability conditions.

*Remark 7:* When the polynomial matrix  $\mathbf{X}(\mathbf{x})$  is chosen as  $\mathbf{X}(\tilde{\mathbf{x}})$  in the first step of the two-step procedure where  $\tilde{\mathbf{x}}$  is defined in Remark 4, the second step can be skipped in the solution searching process. In this case, as the term  $\mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1} = \mathbf{0}$  in  $\mathbf{Q}_{ij}(\mathbf{x})$  for all  $k$ , if there exists a feasible solution in the first step,  $\mathbf{P}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^{-1}$  will be the solution in the second step.

A procedure is given below to apply the theorems developed above to design a stable polynomial fuzzy controller for a nonlinear system.

- 1) Represent the nonlinear system as a polynomial fuzzy model in the form of (2).
- 2) Choose the number of rules and membership functions for the polynomial fuzzy controller in the form of (4).
- 3) Apply the corresponding theorem (Case 1, 2 or 3) according to the chosen number of rules and membership functions of the polynomial fuzzy controller.
- 4) Determine the system stability using step 1 in the theorem by ignoring the term  $\mathbf{B}_i^k(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\mathbf{x})^{-1}$ . If there is no feasible solution, go to step 2, otherwise, next step.
- 5) Determine the system stability using the solution obtained in step 4). If there exists a feasible solution, the PFMB control system is guaranteed to be stable, otherwise, go to step 2).

#### IV. SIMULATION EXAMPLES

Two simulation examples are given to illustrate the merits of the proposed PFMB control approach. The first example is a numerical example which investigates the size of the stability region corresponding to different cases, demonstrating how the number of rules and the shape of membership functions influence the stability analysis results. Also, time-response simulations were performed to verify the investigation. In the second example, we consider an inverted pendulum showing that the proposed PFMB control approach can handle well a benchmark nonlinear system.

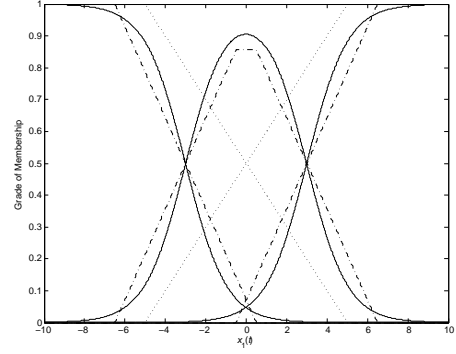


Fig. 1. Membership functions of the polynomial fuzzy model (solid lines) and polynomial fuzzy controller for Case 1 (solid lines), Case 2 (dash-dotted lines) and Case 3 (dotted lines).

##### A. Example 1: Numerical Example

In the first step of the design procedure, we consider that a nonlinear plant is represented by a 3-rule polynomial fuzzy model in the form of (2) with the following parameters:  $\hat{\mathbf{x}} = \mathbf{x} = [x_1 \ x_2]^T$ ,

$$\begin{aligned} \mathbf{A}_1(x_1) &= \begin{bmatrix} 1.59 + 2.45x_1 & -7.29 - 0.89x_1 \\ 0.01 & -0.1 - 0.27x_1^2 \end{bmatrix}, \\ \mathbf{A}_2(x_1) &= \begin{bmatrix} 0.02 - 7.26x_1 - 0.05x_1^2 & -4.64x_1 \\ 0.35 - 0.28x_1 & -0.21 - 1.65x_1^2 \end{bmatrix}, \\ \mathbf{A}_3(x_1) &= \begin{bmatrix} -a + 0.37x_1 - 2.7x_1^2 & -4.33 - 2.73x_1^2 \\ 1.77x_1 & 0.05 - x_1^2 \end{bmatrix}, \\ \mathbf{B}_1(x_1) &= \begin{bmatrix} 1 + 0.37x_1 + 1.28x_1^2 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2(x_1) = \begin{bmatrix} 8 + 0.23x_1^2 \\ 0 \end{bmatrix}, \quad \mathbf{B}_3(x_1) = \begin{bmatrix} -b + 6 + 0.72x_1 + 1.55x_1^2 \\ -1 \end{bmatrix} \end{aligned}$$

where  $a$  and  $b$  are constant scalars.

By applying the second design step of the procedure, we choose the membership functions of the polynomial fuzzy model as  $w_1(x_1) = \mu_{M_1^1}(x_1) = 1 - \frac{1}{1+e^{-(x_1+3)}}$ ,  $w_2(x_1) = \mu_{M_1^2}(x_1) = 1 - w_1(x_1) - w_3(x_1)$ ,  $w_3(x_1) = \mu_{M_1^3}(x_1) = \frac{1}{1+e^{-(x_1-3)}}$ , which are shown in Fig. 1.

In the following, we shall consider all three cases of polynomial fuzzy controllers described in Section III to stabilize the nonlinear plant represented by the polynomial fuzzy model. The stability region of the PFMB control system for each case is investigated with the consideration of the parameters  $2 \leq a \leq 18$  and  $2 \leq b \leq 25$  both at the interval of 1. The stability of the three PFMB control systems is determined by the two-step SOS-based stability conditions in Theorem 1 to Theorem 3.

Based on the membership functions of the polynomial fuzzy model, it is found numerically that  $\gamma_{ij} = 0$  for  $i, j = 1, 2, 3$ ;  $\bar{\gamma}_{11} = \bar{\gamma}_{33} = 1.0000$ ,  $\bar{\gamma}_{12} = \bar{\gamma}_{21} = \bar{\gamma}_{23} = \bar{\gamma}_{32} = 2.4877 \times 10^{-2}$ ,  $\bar{\gamma}_{13} = \bar{\gamma}_{31} = 2.2492 \times 10^{-3}$ ,  $\bar{\gamma}_{22} = 8.1929 \times 10^{-1}$ . We choose  $\varepsilon_1(\mathbf{x}) = \varepsilon_2(\mathbf{x}) = \varphi_1(\mathbf{x}) = \varphi_2(\mathbf{x}) = \eta(\mathbf{x}) = 0.0001$ ; the degrees of  $\mathbf{X}(x_1)$ ,  $\mathbf{N}_j(x_1)$  and  $\mathbf{P}(x_1)$  are 0, 0 and 2, respectively, for all three cases. The third-party Matlab toolbox SOSTOOLS [71] is used to search for a feasible solution to the two-step SOS-based stability conditions. In the following, step

3) to step 5) in the design procedure are employed to design the polynomial fuzzy controller and determine the PFMB control system using Theorem 1 to Theorem 3.

1) *Case 1:* We consider a polynomial fuzzy controller sharing the same number of rules and premise membership functions as those of the polynomial fuzzy model, i.e.,  $c = p$  and  $m_i = w_i$ ,  $i = 1, \dots, p$  discussed in Section III-A. The two-step SOS-based stability conditions in Theorem 1 are employed to check the stability of the PFMB control system. The stability region is shown in Fig. 2 indicated by ‘o’.

2) *Case 2:* We consider a polynomial fuzzy controller sharing the same number of rules as those of the polynomial fuzzy model but not the premise membership functions, i.e.,  $c = p$  and  $m_i \neq w_i$  for any  $i$  discussed in Section III-B. It is chosen for demonstration purposes that the membership functions of the polynomial fuzzy controller as  $m_1(x_1) =$

$$\mu_{N_1^1}(x_1) = \begin{cases} 1 & \text{for } x_1 < -6.5 \\ \frac{-x_1+0.5}{7} & \text{for } -6.5 \leq x_1 \leq 0.5 \\ 0 & \text{for } x_1 > 0.5 \end{cases},$$

$$m_2(x_1) = \mu_{N_1^2}(x_1) = 1 - m_1(x_1) - m_3(x_1) \text{ and } m_3(x_1) =$$

$$\mu_{N_1^3}(x_1) = \begin{cases} 0 & \text{for } x_1 < -0.5 \\ \frac{x_1+0.5}{7} & \text{for } -0.5 \leq x_1 \leq 6.5 \\ 1 & \text{for } x_1 > 6.5 \end{cases}, \text{ which are}$$

shown in Fig. 1.

With the chosen membership functions, it is found numerically that  $\underline{\sigma}_1 = \underline{\sigma}_3 = -1.0350 \times 10^{-1}$  and  $\underline{\sigma}_2 = -9.2401 \times 10^{-2}$ ;  $\bar{\sigma}_1 = \bar{\sigma}_3 = 1.0350 \times 10^{-1}$  and  $\bar{\sigma}_2 = 1.0402 \times 10^{-1}$ . The two-step SOS-based stability conditions in Theorem 2 are employed to check the stability of the PFMB control system. The stability region is shown in Fig. 2 indicated by ‘□’.

3) *Case 3:* We consider a polynomial fuzzy model and polynomial fuzzy controller employing different number of rules and premise membership functions, i.e.,  $c \neq p$  as discussed in Section III-C. A 2-rule polynomial fuzzy controller is employed to control the nonlinear plant represented by the polynomial fuzzy model. The membership functions are chosen as  $m_1(x_1) = \mu_{N_1^1}(x_1) =$

$$\begin{cases} 1 & \text{for } x_1 < -5 \\ \frac{-x_1+5}{10} & \text{for } -5 \leq x_1 \leq 5 \\ 0 & \text{for } x_1 > 5 \end{cases} \text{ and } m_2(x_1) = \mu_{N_1^2}(x_1) =$$

$$1 - m_1(x_1), \text{ which are shown in Fig. 1. The two-step SOS-based stability conditions in Theorem 3 are employed to check the stability of the PFMB control system. The stability region is shown in Fig. 2 indicated by ‘x’.$$

Referring to Fig. 2, it can be seen that Case 1 produces the largest size of stability region while Case 2 comes second and Case 3 produces the smallest size. The result is reasonable that Case 1 taking the advantage of the perfectly matched number of rules and membership functions in favor of the stability analysis is able to produce the largest size of stability region compared with Case 2 and Case 3. Although Case 2 cannot outperform Case 1 in terms of the size of stability region, its size is still larger than that of Case 3 by taking the advantage of matched number of rules. Among the three cases, Case 3 offers the lowest controller complexity (when a

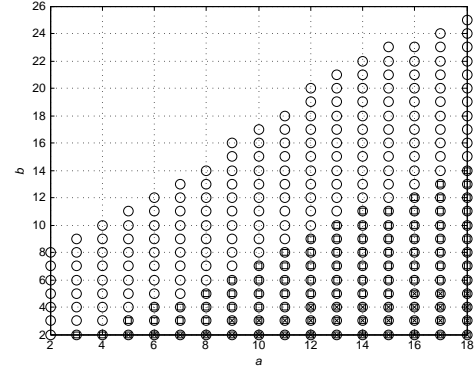


Fig. 2. Stability regions given by Theorem 1 (‘o’), Theorem 2 (‘□’) and Theorem 3 (‘x’).

smaller number of rules and/or simple membership functions are used) and greatest design flexibility while Case 2 comes second and Cases 3 the last. During the control design, it suggests that Case 3 should be employed at the beginning in order to achieve a polynomial fuzzy controller with the lowest implementation cost. If a stable design cannot be achieved, Case 2 can be employed then followed by Case 1.

For comparison purposes, the SOS-based stability conditions in [67], [69], which can only be applied to Case 1 and use  $\mathbf{X}(\mathbf{x})$  of degree 0 (because there is no entire zero row in  $\mathbf{B}_j(x_1)$  for all  $j$ ), are employed to check the system stability. However, no stability region can be found. In this paper, unlike the published work, there is no constraint on both  $\mathbf{X}(\mathbf{x})$  and  $\mathbf{P}(\mathbf{x})$ , which can be polynomial matrices in any state variables independent of the form of the input matrices  $\mathbf{B}_j(\mathbf{x})$ . It can be seen from this example that elimination of this constraint is able to produce a more relaxed stability analysis result.

To verify the stability analysis result, the time responses of PFMB control systems for the three cases were simulated. Referring to Fig. 2, considering  $a = 18$  and  $b = 25$  in Case 1 (stability region indicated by ‘o’), we obtained  $\mathbf{P}(x_1) = \begin{bmatrix} P_{11}(x_1) & P_{12}(x_1) \\ P_{21}(x_1) & P_{22}(x_1) \end{bmatrix}$  where  $P_{11}(x_1) = 0.1827 - 0.0337x_1 + 0.0134x_1^2$ ,  $P_{12}(x_1) = P_{21}(x_1) = 0.0451 - 0.0075x_1 + 0.0047x_1^2$  and  $P_{22}(x_1) = 2.5034 - 0.0260x_1 + 0.3169x_1^2$ ; the feedback gains as  $\mathbf{G}_j(x_1) = [G_{j1}(x_1) \ G_{j2}(x_1)]$  where  $G_{11}(x_1) = -5.0914 - 2.3612x_1 - 0.7706x_1^2$ ,  $G_{12}(x_1) = 2.3194 - 0.8255x_1 - 0.2425x_1^2$ ,  $G_{21}(x_1) = -0.7653 + 0.0709x_1 - 0.2788x_1^2$ ,  $G_{22}(x_1) = 0.2101 + 0.6914x_1 - 0.1081x_1^2$ ,  $G_{31}(x_1) = -0.7004 + 0.1762x_1 - 0.2346x_1^2$  and  $G_{32}(x_1) = 0.4710 + 0.9222x_1 - 0.0669x_1^2$ .

In Case 2 (stability region indicated by ‘□’), considering  $a = 18$  and  $b = 14$ , we obtained  $P_{11}(x_1) = 30.0790 + 10.0390x_1 + 12.1192x_1^2$ ,  $P_{12}(x_1) = P_{21}(x_1) = 11.9706 + 11.2980x_1 - 2.4291x_1^2$  and  $P_{22}(x_1) = 748.7599 + 296.4584x_1 + 215.2665x_1^2$ ; the feedback gains as  $G_{11}(x_1) = -4.5457 + 0.0230x_1 - 0.5121x_1^2$ ,  $G_{12}(x_1) = 1.8860 + 0.2921x_1 - 0.2555x_1^2$ ,  $G_{21}(x_1) = -1.0547 + 0.5044x_1 - 0.3989x_1^2$ ,  $G_{22}(x_1) = 0.4211 + 0.6804x_1 - 0.0547x_1^2$ ,  $G_{31}(x_1) = -1.0554 + 0.6516x_1 - 0.2546x_1^2$  and  $G_{32}(x_1) =$

$$0.4364 + 0.6029x_1 + 0.0028x_1^2.$$

In Case 3 (stability region indicated by 'x'), considering  $a = 18$  and  $b = 5$ , we obtained  $P_{11}(x_1) = 0.7864 - 0.3906x_1 + 0.4747x_1^2$ ,  $P_{12}(x_1) = P_{21}(x_1) = 0.3902 - 0.1299x_1 + 0.1917x_1^2$  and  $P_{22}(x_1) = 9.0285 + 1.2287x_1 + 4.8239x_1^2$ ; the feedback gains as  $G_{11}(x_1) = -6.5643 + 0.1984x_1 - 0.8689x_1^2$ ,  $G_{12}(x_1) = -0.2489 - 0.3929x_1 - 0.4573x_1^2$ ,  $G_{21}(x_1) = -1.4034 + 1.0396x_1 - 0.5679x_1^2$  and  $G_{22}(x_1) = 0.1505 + 0.2292x_1 - 0.2254x_1^2$ .

The phase plots of  $x_1$  and  $x_2$  for the three cases subject to various initial conditions are shown in Fig. 3 to Fig. 5. It can be seen that the PFMB control system are all stable and the polynomial fuzzy controllers are able to drive the system states to the origin.

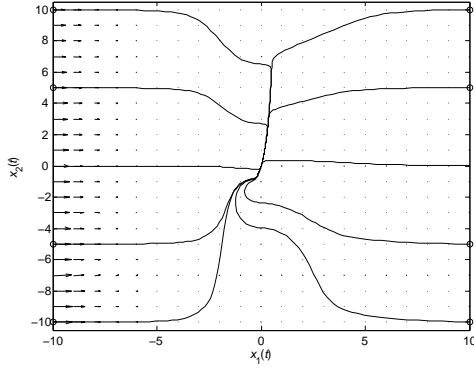


Fig. 3. Phase plot of  $x_1(t)$  and  $x_2(t)$  of PFMB control system for Cases 1 with  $a = 18$  and  $b = 25$  where the initial conditions are indicated by 'o'.

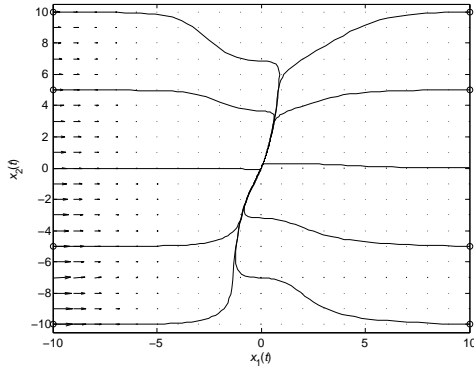


Fig. 4. Phase plot of  $x_1(t)$  and  $x_2(t)$  of PFMB control system for Cases 2 with  $a = 18$  and  $b = 14$  where the initial conditions are indicated by 'o'.

### B. Example 2: Inverted Pendulum

An inverted pendulum on a cart is considered as a nonlinear plant [47] where the system dynamics is described by the following state-space equations.

$$\dot{x}_1(t) = x_2(t) \quad (58)$$

$$\dot{x}_2(t) = \frac{g \sin(x_1(t)) - am_p L x_2(t)^2 \sin(x_1(t)) \cos(x_1(t))}{4L/3 - am_p L \cos^2(x_1(t))} - \frac{a \cos(x_1(t)) u(t)}{4L/3 - am_p L \cos^2(x_1(t))} \quad (59)$$

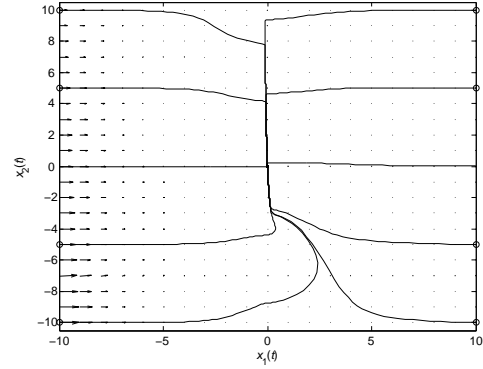


Fig. 5. Phase plot of  $x_1(t)$  and  $x_2(t)$  of PFMB control system for Cases 3 with  $a = 18$  and  $b = 5$  where the initial conditions are indicated by 'o'.

where  $x_1(t)$  and  $x_2(t)$  are the angular displacement and angular velocity of the pendulum, respectively,  $g = 9.8m/s^2$  is the acceleration due to gravity,  $m_p = 2kg$  is the mass of the pendulum,  $M_c = 8kg$  is the mass of the cart,  $a = 1/(m_p + M_c)$ ,  $2L = 1m$  is the length of the pendulum, and  $u(t)$  is the force applied to the cart.

It is reported in [76] that the inverted pendulum working in the operating domain of  $x_1(t) \in [-\frac{5\pi}{12}, \frac{5\pi}{12}]$  can be described by a 2-rule polynomial fuzzy model with

$$\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x}, \mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ a_1 & 0 \end{bmatrix}, \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ a_2 & 0 \end{bmatrix},$$

$$\mathbf{B}_1(\mathbf{x}) = \begin{bmatrix} 0 \\ -f_{1_{min}} a \end{bmatrix}, \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} 0 \\ -f_{1_{max}} a \end{bmatrix}, a_1 = f_{1_{min}} (g(t_3 x_1(t)^2 + t_1) - am_p L x_2(t)^2 (s_3 x_1(t)^2 + s_1)),$$

$$a_2 = f_{1_{max}} (g(t_3 x_1(t)^2 + t_1) - am_p L x_2(t)^2 (s_3 x_1(t)^2 + s_1))$$

where  $f_{1_{min}} = 0.3922$ ,  $f_{1_{max}} = 1.7647$ ,  $s_3 = -0.1460$ ,  $s_1 = 0.9897$ ,  $t_3 = 1.0545$  and  $t_1 = 0.6469$ ; the membership functions are given as  $\mu_{M_1^1}(x_1(t)) = w_1(x_1(t)) = \frac{f_1(x_1(t)) - f_{1_{max}}}{f_{1_{min}} - f_{1_{max}}}$  and  $\mu_{M_2^1}(x_1(t)) = w_2(x_1(t)) = 1 - \mu_{M_1^1}(x_1(t))$ .

It is found numerically that  $\gamma_{ij} = 0$  for  $i, j = 1, 2$ ;  $\bar{\gamma}_{11} = \bar{\gamma}_{22} = 1.0000$ ,  $\bar{\gamma}_{12} = \bar{\gamma}_{21} = 0.25$ . The two-step SOS-based stability conditions in Theorem 1 are employed to design a polynomial fuzzy controller based on the 2-rule polynomial fuzzy model. We choose  $\varepsilon_1(\mathbf{x}) = \varepsilon_2(\mathbf{x}) = \varphi_1(\mathbf{x}) = \varphi_2(\mathbf{x}) = \eta(\mathbf{x}) = 0.0001$ . The degrees of  $\mathbf{X}(x_2)$ ,  $\mathbf{N}(x_1)$  and  $\mathbf{P}(x_2)$  are all 2. By using the third-party Matlab toolbox SOSTOOLS

[71], we obtained  $\mathbf{P}(x_2) = \begin{bmatrix} P_{11}(x_2) & P_{12}(x_2) \\ P_{21}(x_2) & P_{22}(x_2) \end{bmatrix}$  where

$$P_{11}(x_2) = 19.4196 - 4.6787x_2 + 4.0607x_2^2, P_{12}(x_2) = P_{21}(x_2) = 2.8551 - 3.7377x_2 + 0.7072x_2^2$$

$$\text{and } P_{22}(x_2) = 2.1824 - 0.6394x_2 + 0.2513x_2^2;$$

the feedback gains as  $\mathbf{G}_j(\mathbf{x}) = \begin{bmatrix} \frac{G_{j1}(\mathbf{x})}{h(x_2)} & \frac{G_{j2}(\mathbf{x})}{h(x_2)} \end{bmatrix}$  where  $G_{11}(\mathbf{x}) = 1.1541x_1^2x_2^2 - 7.1291 \times 10^{-5}x_1^2x_2 + 1443.0073x_1^2 - 1.7097 \times 10^{-7}x_1x_2^2 + 1.0534 \times 10^{-11}x_1x_2 - 1.8327 \times 10^{-4}x_1 + 1.4071x_2^2 - 8.7046 \times 10^{-5}x_2 + 1898.9641$ ,  $G_{12}(\mathbf{x}) = 0.0763x_1^2x_2^2 - 4.3844 \times 10^{-6}x_1^2x_2 + 461.4913x_1^2 - 9.1876 \times 10^{-9}x_1x_2^2 + 7.0839 \times 10^{-13}x_1x_2 - 5.7692 \times 10^{-5}x_1 + 0.1027x_2^2 - 5.0762 \times 10^{-6}x_2 + 611.5159$ ,  $G_{21}(\mathbf{x}) = 1.1161x_1^2x_2^2 - 6.8676 \times 10^{-5}x_1^2x_2 + 1097.0242x_1^2 - 3.0531 \times 10^{-7}x_1x_2^2 + 1.8817 \times 10^{-11}x_1x_2 - 3.3417 \times 10^{-4}x_1 + 0.8312x_2^2 - 5.1258 \times 10^{-5}x_2 + 942.5031$ ,

$$G_{22}(\mathbf{x}) = 0.0531x_1^2x_2^2 - 4.8163 \times 10^{-6}x_1^2x_2 + 341.8519x_1^2 - 1.6885 \times 10^{-8}x_1x_2^2 + 1.2517 \times 10^{-12}x_1x_2 - 1.0544 \times 10^{-4}x_1 + 0.0482x_2^2 - 3.3446 \times 10^{-6}x_2 + 298.5089 \text{ and } h(x_2) = 1.8399 \times 10^{-6}x_2^4 - 6.08102 \times 10^{-11}x_2^3 + 9.8376 \times 10^{-3}x_2^2 - 3.9949 \times 10^{-7}x_2 + 1.$$

The polynomial fuzzy controller is employed to control the inverted pendulum with the initial conditions of  $\mathbf{x}(0) = [\frac{5\pi}{12} \ 0]^T$ ,  $\mathbf{x}(0) = [\frac{5\pi}{24} \ 0]^T$ ,  $\mathbf{x}(0) = [-\frac{5\pi}{24} \ 0]^T$  and  $\mathbf{x}(0) = [-\frac{5\pi}{12} \ 0]^T$ . The system responses are shown in Fig. 6 and Fig. 7. It can be seen that the inverted pendulum can be stabilized by the polynomial fuzzy controller.

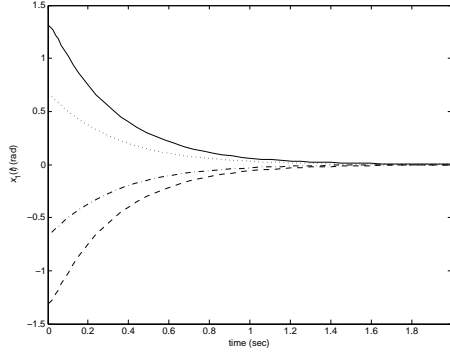


Fig. 6. System responses of  $x_1(t)$  of the inverted pendulum with the initial conditions  $\mathbf{x}(0) = [\frac{5\pi}{12} \ 0]^T$  (solid line),  $\mathbf{x}(0) = [\frac{5\pi}{24} \ 0]^T$  (dotted line),  $\mathbf{x}(0) = [-\frac{5\pi}{24} \ 0]^T$  (dash-dot line) and  $\mathbf{x}(0) = [-\frac{5\pi}{12} \ 0]^T$  (dashed line).

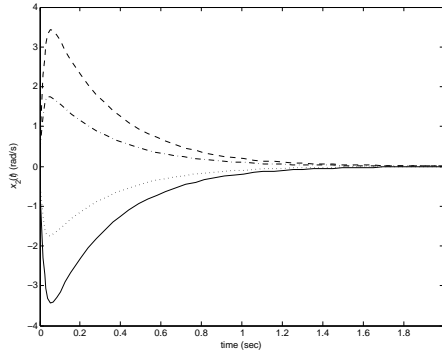


Fig. 7. System responses of  $x_2(t)$  of the inverted pendulum with the initial conditions  $\mathbf{x}(0) = [\frac{5\pi}{12} \ 0]^T$  (solid line),  $\mathbf{x}(0) = [\frac{5\pi}{24} \ 0]^T$  (dotted line),  $\mathbf{x}(0) = [-\frac{5\pi}{24} \ 0]^T$  (dash-dot line) and  $\mathbf{x}(0) = [-\frac{5\pi}{12} \ 0]^T$  (dashed line).

## V. CONCLUSION

This paper has investigated the stability of PFMB control systems based on the SOS-based approach with the support of the Lyapunov stability theory. Three cases of polynomial fuzzy controllers with the consideration of matched/mismatched number of rules and/or premise membership functions, which offer various levels of controller complexity, design flexibility and stability analysis results, have been investigated. A polynomial Lyapunov function candidate independent of the form of the polynomial fuzzy model such that its polynomial matrix being allowed in any state variables has been proposed for stability analysis. SOS-based stability conditions have been

obtained with the consideration of boundary information of membership functions and a two-step procedure has been proposed to find numerically a feasible solution. Without the constraint on the polynomial matrix of the polynomial Lyapunov function candidate, the polynomial fuzzy control approach can be applied to a wider class of nonlinear plants and the solution space is enlarged resulting in a more relaxed stability analysis result. Simulation examples have been given to demonstrate the merits of the proposed PFMB control approach.

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**H.K. Lam** (M’98-SM’10) received the B.Eng. (Hons.) and Ph.D. degrees from the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hong Kong, in 1995 and 2000, respectively.

From 2000 to 2005, he was a Postdoctoral Fellow and a Research Fellow with the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, respectively. In 2005, he joined Kings College London, London, U.K., as a Lecturer and currently is a senior lecturer.

His current research interests include intelligent control systems and computational intelligence. He has served as a program committee member and international advisory board member for various international conferences and a reviewer for various books, international journals and international conferences. He is an associate editor for *IEEE Transactions on Fuzzy Systems*, *International Journal of Fuzzy Systems* and *Journal of Intelligent Learning Systems and Applications*; and guest editor for a number of international journals. He is in the editorial boards of a number of journals including *IET Control Theory & Applications*.

He is the coeditor for two edited volumes: *Control of Chaotic Nonlinear Circuits* (World Scientific, 2009) and *Computational Intelligence and Its Applications* (World Scientific, 2012), and the coauthor of the book *Stability Analysis of Fuzzy-Model-Based Control Systems* (Springer, 2011).



**Ligang Wu** (SM’12) received the B.S. degree in Automation from Harbin University of Science and Technology, China in 2001; the M.E. degree in Navigation Guidance and Control from Harbin Institute of Technology, China in 2003; the PhD degree in Control Theory and Control Engineering from Harbin Institute of Technology, China in 2006. From January 2006 to April 2007, he was a Research Associate in the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong. From September 2007 to June 2008, he was a Senior

Research Associate in the Department of Mathematics, City University of Hong Kong, Hong Kong. From December 2012 to December 2013, he was a Research Associate in the Department of Electrical and Electronic Engineering, Imperial College London, London, UK. In 2008, he joined the Harbin Institute of Technology, China, as an Associate Professor, and was then promoted to a Professor in 2012.

Prof. Wu currently serves as an Associate Editor for a number of journals, including *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, *IEEE ACCESS*, *Information Sciences*, *Signal Processing*, *IET Control Theory and Applications*, and *Circuits Systems and Signal Processing*. He is also an Associate Editor for the Conference Editorial Board, *IEEE Control Systems Society*. His current research interests include complex hybrid dynamical systems, optimal filtering, multi-objective control and optimization techniques, and model reduction.



**James Lam** (S’86, M’88, SM’99, F’12) received a BSc (1<sup>st</sup> Hons.) degree in Mechanical Engineering from the University of Manchester, and was awarded the Ashbury Scholarship, the A.H. Gibson Prize, and the H. Wright Baker Prize for his academic performance. He obtained the MPhil and PhD degrees from the University of Cambridge. He is a recipient of the Croucher Foundation Scholarship and Fellowship, the Outstanding Researcher Award of the University of Hong Kong, and the Distinguished Visiting Fellowship of the Royal Academy of Engineering. Prior to joining the University of Hong Kong in 1993 where he is now Chair Professor of Control Engineering, Professor Lam held lectureships at the City University of Hong Kong and the University of Melbourne.

Professor Lam is a Chartered Mathematician, Chartered Scientist, Chartered Engineer, Fellow of Institute of Electrical and Electronic Engineers, Fellow of Institution of Engineering and Technology, Fellow of Institution of Mathematics and Its Applications, and Fellow of Institution of Mechanical Engineers. He is Editor-in-Chief of *IET Control Theory and Applications* and *Journal of The Franklin Institute*, Subject Editor of *Journal of Sound and Vibration*, Editor of *Asian Journal of Control*, Section Editor of *Cogent Engineering*, Associate Editor of *Automatica*, *International Journal of Systems Science*, *International Journal of Applied Mathematics and Computer Science*, *Multidimensional Systems and Signal Processing*, and *Proc. IMechE Part I: Journal of Systems and Control Engineering*. He is a member of the IFAC Technical Committee on Networked Systems. His research interests include model reduction, robust synthesis, delay, singular systems, stochastic systems, multidimensional systems, networked control systems and vibration control.